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Resonances of a conducting drop in an alternating electric field

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Resonance phenomena of a conducting drop forced by an alternating electric field are studied by perturbation analysis. Although the motions are assumed to be irrotational, weak viscous effects are included in the boundary condition of the normal stress balance. Without an external field, the first-order expansion of the domain perturbations yields the same result as that obtained by Lamb (1932) for the viscous decay of the free oscillation modes. A primary resonance occurs in the first-order forced oscillation problem. Under strong excitation, superharmonic, subharmonic, and coupled resonances are revealed in the second-order solutions. Hence, large-amplitude drop oscillations may occur even if the excitation frequencies are away from the characteristic drop frequencies and the spatial forms of the excitation modes do not directly match the drop shape modes. In order to obtain comparable response amplitudes, however, the magnitudes of external forcing required to excite secondary resonances are shown to be about an order greater than that for the primary resonances.

1. Introduction

The consideration of drop oscillations is important in a variety of scientific and technical problems. Natural raindrops are often in a state of oscillation as seen from *in situ* photographs (Jones 1959). Possible causes of such oscillations have been proposed, such as collisions among drops (Beard, Johnson & Jameson 1983) as well as resonances with vortex shedding in drop wakes (Gunn 1949; Beard, Ochs & Kubesh 1989) and eddies in a turbulent air (Blanchard 1950). There remain, however, basic questions about how the drop resonances occur in response to the external excitations with various frequencies and spatial forms that exist in natural systems. In crystal growth processes, oscillations of free liquid surfaces are undesirable since they can cause crystal diameter perturbations, and the accompanying internal fluid flows will also influence the growth segregation behaviour (Carruthers 1974). Hence, knowledge of the resonant responses in drops to the external vibrations should be also helpful in apparatus design for materials processing.

In many experimental studies of drop oscillations, external excitation forces have to be employed to maintain observably large oscillatory amplitudes against viscous damping (Trinh & Wang 1982). An effective way to excite large-amplitude oscillations is to tune the frequency of the external excitation force so that it is the

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same as the characteristic drop frequency of the desired mode (primary resonances). The problem specifically concerning the secondary resonances which may occur under strong excitation due to the nonlinearities in drop dynamics still remains unexplored. The studies of the nonlinear internal resonances of unforced drops presented by Tsamopoulos & Brown (1984) and Naterajan & Brown (1986, 1987) give valuable insight in understanding the complex interactions between the free oscillation modes, yet no information has been given about the secondary resonances when a drop is forced by external excitations. We found no direct reference in the literature to the problem of nonlinear forced drop resonances, although drop oscillations have been intensively studied for over a century (e.g. Rayleigh 1879; Lamb 1932; Brazier-Smith *et al.* 1971; Marston 1980; Tsamopoulos & Brown 1983, 1984; Beard 1984; Naterajan & Brown 1986, 1987; Lundgren & Mansour 1988; etc.). Instead of treating the general problem of forced drop resonances with its formidable algebra and poorly understood details about various natural forces, in this work we present a theoretical analysis of some nonlinear resonances occurring when a conducting drop oscillates in response to an alternating electric field. The findings of this specific study, however, may provide some insight into drop oscillations excited by other types of external forces.

An analytical method of domain perturbations (Joseph 1973; Tsamopoulos & Brown 1983, 1984) is utilized to deal with the nonlinear aspects of drop resonances. For most cases where detailed studies are presented in this paper, the order of magnitude of the drop shape deformations is presumed to be the same as that of the external forcing term arising from the alternating electric field. Thus, with respect to a small parameter serving as a measure of the magnitude of drop shape deformations, the first-order expansion leads to solutions similar to forced linear oscillators. In carrying out the expansion to the second order, various secondary resonances may appear under the condition of strong excitation. By investigating the solvability conditions, which are posed to eliminate so-called secular and small-divisor terms, we may gain further information about the first-order oscillation terms which can differ significantly from the linear results.

In order to make use of the domain perturbation technique, we confine the analysis to considering very weak viscous effects which are believed to be significant only within a thin vortical surface layer so that the motions elsewhere in the drop may be reasonably assumed irrotational. The viscous effects are incorporated by a novel method of formulating a normal damping stress term in the boundary condition at the free surface based on an assumption that the overall rate of work done by this damping stress equals the total rate of dissipation of mechanical energy as given by Lamb (1932, §329; also presented in Batchelor 1967, §4.1). Thus the derivations in this problem deal completely with potential flow so that the complicated manipulation of the boundary-layer equations for the weak vortical flow can be avoided. Since the field equation governing the irrotational flow is the Laplace equation, modifying the boundary conditions at the surface should be an acceptable means of including the small viscous effects.

This paper starts in §2 with the mathematical formulation of the governing equations and an outline of the perturbation scheme. A detailed derivation of the expression for the viscous normal stress that appears in the boundary condition at the free surface is presented in §3. The results of the first-order expansion are discussed in §4. The solutions that manifest secondary resonances are studied in §5 where second-order expansions are carried out. A brief summary of the results is presented in §6.

2. Formulation

We consider the irrotational and incompressible motion of an electrically conducting drop with volume $\frac{4}{3}\pi R^3$, density ρ, uniform interfacial tension σ and zero electric charge. The motion is under the influences of an externally applied uniform electric field with a pure a.c. component, $E_a^* \cos \Omega^* t^*$, and a slight viscous damping. For simplicity, only the axisymmetric case is involved in this study, where the axis of symmetry is parallel to the direction of the external electric field. With an asterisk denoting dimensional variables, we define the corresponding dimensionless variables: radial coordinate $r_0 = r^*/R$, time $t = t^*[\sigma/(\rho R^3)]^{\frac{1}{2}}$, velocity potential $\Phi = \Phi^*[\rho/(\sigma R)]^{\frac{1}{2}}$, electric potential $V = V^*/(E_a^* R)$, electric field $E = E^*(\epsilon_m R/\sigma)^{\frac{1}{2}}$, and normal stress terms such as pressure, electric stress and viscous stress $(\Delta p_0, p, N_e, N_d) = (R/\sigma)(\Delta p_0^*, p^*, N_e^*, N_d^*)$, where ϵ_m (in SI units) is the permittivity of the surrounding medium, which is assumed to be a tenuous insulating gas so that both its hydrodynamical and electrodynamical effects may be ignored. Thus, in dimensionless forms, the governing equations for the velocity potential Φ and for the electric potential V are

$$\nabla^2 \Phi = 0 \quad (0 \leqslant r_0 \leqslant F(\theta, t)), \tag{2.1}$$

$$\nabla^2 V = 0 \quad (F(\theta, t) \leqslant r_0 \leqslant \infty), \tag{2.2}$$

along with the boundary conditions

$$\frac{\partial \Phi}{\partial r_0} \neq \infty \quad (r_0 = 0), \tag{2.3}$$

$$V = -r_0 \cos\theta \cos \Omega t \quad (r_0 \to \infty), \tag{2.4}$$

$$\frac{\partial \Phi}{\partial r_0} = \frac{\partial F}{\partial t} + \frac{1}{r_0^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial F}{\partial \theta} \quad (r_0 = F(\theta, t)), \tag{2.5}$$

$$\Delta p_0 - \frac{\partial \Phi}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial r_0}\right)^2 + \left(\frac{1}{r_0}\frac{\partial \Phi}{\partial \theta}\right)^2\right] + N_e + N_d = \nabla \cdot \boldsymbol{n} \quad (r_0 = F(\theta, t)), \tag{2.6}$$

$$\int_0^\pi (\boldsymbol{n} \cdot \nabla V)_{r_0 = F}[F^2 + (\partial F/\partial \theta)^2]^{\frac{1}{2}} F \sin\theta \, \mathrm{d}\theta = 0, \tag{2.7}$$

$$\boldsymbol{t} \cdot \nabla V = 0 \quad (r_0 = F(\theta, t)), \tag{2.8}$$

where the unit normal and tangential vectors of the drop surface can be written as

$$\boldsymbol{n} = \frac{F \boldsymbol{e}_r - (\partial F/\partial \theta)\, \boldsymbol{e}_\theta}{[F^2 + (\partial F/\partial \theta)^2]^{\frac{1}{2}}}, \quad \boldsymbol{t} = \frac{(\partial F/\partial \theta)\, \boldsymbol{e}_r + F \boldsymbol{e}_\theta}{[F^2 + (\partial F/\partial \theta)^2]^{\frac{1}{2}}}. \tag{2.9}$$

The surface of the drop is described by $RF(\theta, t)$, where $F(\theta, t)$ is the dimensionless shape function of the drop and θ is the meridional angle in spherical coordinates measured from the axis of symmetry. The condition (2.3) ensures that the velocity is finite at the origin at the centre of mass of the drop, while (2.4) is the far-field condition for the electric potential. Equation (2.5) gives the kinematic relation between the motion of the drop surface and the local velocity field. The condition (2.6) is a normal stress balance at the interface, where the pressure differences caused by capillarity and drop motion (from Bernoulli's equation) are equated to the normal electric stress N_e, having the dimensionless form (Landau & Lifshitz 1959)

$$N_e = \frac{\epsilon_m R E_a^{*2}}{2\sigma}\left[\left(\frac{\partial V}{\partial r_0}\right)^2 + \left(\frac{1}{r_0}\frac{\partial V}{\partial \theta}\right)^2\right]_{r_0 = F(\theta, t)}, \tag{2.10}$$

and a damping stress N_d which will be derived in detail later in §3. Equation (2.7) represents the conservation of electric charge in a conducting drop, and (2.8) guarantees that the tangential component of the electric field is continuous across the interface.

In addition, the solution for drop shape must satisfy the constraint for constant volume of the drop

$$\int_0^\pi F^3(\theta,t)\sin\theta\,\mathrm{d}\theta = 2, \tag{2.11}$$

and the constraint that the centre of mass of the drop remains at the origin

$$\int_0^\pi F^4(\theta,t)\cos\theta\sin\theta\,\mathrm{d}\theta = 0, \tag{2.12}$$

which physically results from Newton's law of motion for the case of zero net force on the drop.

The exact solution to the problem (2.1)–(2.12) is intractable because of the nonlinearities and an unknown domain on which it is posed. However, for a nearly spherical drop, it is convenient to use the domain perturbation technique to transform the drop shape into a unit sphere (Tsamopoulos & Brown 1983). This is done by introducing the change of coordinates

$$r_0 \equiv rF(\theta,t). \tag{2.13}$$

Thus the interface $r_0 \equiv F(\theta,t)$ of a complicated configuration is mapped into a simple domain $r=1$. With respect to a small parameter ϵ, used as the scaling of the magnitude of the drop deformation from the spherical shape, we may expand every dependent variable, say a function $f(r_0,\theta,t;\epsilon)$, in a Taylor series as

$$f(r_0,\theta,t;\epsilon) = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}f^{[n]}(r,\theta,t), \tag{2.14}$$

where
$$f^{[n]}(r,\theta,t) \equiv \left[\left(\frac{\partial}{\partial\epsilon}+r\frac{\partial F}{\partial\epsilon}\frac{\partial}{\partial r_0}\right)^n f(r_0,\theta,t;\epsilon)\right]_{\substack{\epsilon=0\\ r_0=r}}. \tag{2.15}$$

Moreover, it is usually convenient to use the notation

$$f^{\langle n\rangle}(r,\theta,t) \equiv \left[\frac{\partial^n f(r_0,\theta,t;\epsilon)}{\partial\epsilon^n}\right]_{\substack{\epsilon=0\\ r_0=r}}, \tag{2.16}$$

because $f^{[n]}(r,\theta,t)$ can always be expressed in terms of $f^{\langle n\rangle}(r,\theta,t)$ as illustrated in previous papers (cf. Tsamopoulos & Brown 1983; Feng 1990; Feng & Beard 1990). As can be seen, $f^{\langle n\rangle}(r,\theta,t)$ denotes the contribution based on the spherical domain $r=1$, whereas $f^{[n]}(r,\theta,t)$ is a sum of $f^{\langle n\rangle}(r,\theta,t)$ and other terms that account for the deformation of the domain which arise from $\partial/\partial r_0$.

Since we are considering the drop deformations caused by the electric surface stress stemming from the alternating external electric field, for most cases studied in this work, it would be convenient to assume an ordering relation

$$\frac{\epsilon_{\mathrm{m}} R E_{\mathrm{a}}^{*2}}{\sigma} = \epsilon K, \tag{2.17}$$

with K as a constant of proportionality. In general, for some resonant problems, perturbation analyses can also be conducted under the assumption of $\epsilon^n K$, where n

is a positive integer, instead of ϵK on the right-hand-side of (2.17). However, higher-order expansion are required to be carried out to reveal the secondary resonances for larger n.

The zeroth-order ($\epsilon = 0$) solution of the system

$$\begin{bmatrix} F^{\langle 0\rangle} \\ \Phi^{\langle 0\rangle} \\ V^{\langle 0\rangle} \\ \Delta p_0^{\langle 0\rangle} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -(r-r^{-2})\cos\theta\cos\Omega t \\ 2 \end{bmatrix} \tag{2.18}$$

recovers, as expected, the static conducting sphere in an electric field which, when scaled by $\epsilon = 0$, has no influence to the drop shape.

The expansion of Laplace equations (2.1) and (2.2) yields

$$\nabla^2\Phi^{\langle n\rangle} = 0 \quad (0 \leqslant r \leqslant 1), \qquad \nabla^2 V^{\langle n\rangle} = 0 \quad (1 \leqslant r < \infty). \tag{2.19}$$

Therefore the solutions for the velocity and electric potentials at each order in ϵ that satisfy the natural boundary conditions (2.3) and (2.4) in terms of domain perturbations at $r = 0$ and at $r \to \infty$ may be written as

$$\begin{bmatrix} \Phi^{\langle n\rangle} \\ V^{\langle n\rangle} \end{bmatrix} = \sum_{l=0}^{\infty} \begin{bmatrix} \beta_l^{\langle n\rangle}(t)\, r^l \\ \xi_l^{\langle n\rangle}(t)\, r^{-l-1} \end{bmatrix} P_l(\theta) \quad (n > 0), \tag{2.20}$$

where $P_l(\theta)$ is the Legendre polynomial of order l. For convenience, the shape function is also expanded at each order in ϵ as

$$F^{\langle n\rangle}(\theta, t) = \sum_{l=0}^{\infty} \alpha_l^{\langle n\rangle}(t)\, P_l(\theta). \tag{2.21}$$

As a result, in higher-order expansions of domain perturbations, most boundary conditions consist of terms from the contribution based on the spherical domain as well as terms that account for the deformation of the interface. Therefore, by adjusting the form of the boundary conditions at each order of ϵ, the solutions can be evaluated on the simple spherical domain.

In order to properly account for the slower time evolution due to the nonlinear interactions among the modes, so that secular and small-divisor terms that may appear in the inhomogeneous problems of higher-order asymptotic expansions can be avoided, it is usually convenient to make use of the method of multiple timescales (Nayfeh & Mook 1979). Thus, the relation

$$\frac{\partial}{\partial t} \equiv \sum_{n=0}^{\infty} \epsilon^n \frac{\partial}{\partial T_n} \quad \text{with} \quad T_n \equiv \epsilon^n t \tag{2.22}$$

will be used in what follows.

3. Viscous damping stress

At a free surface, the physical conditions to be satisfied for the stress are that the tangential component is zero and that the normal component equates the sum of a constant term and any contribution from surface tension (Batchelor 1967). If the viscous forces are very small in comparison with non-viscous forces, then viscosity only produces a thin, weak vortical layer at the free surface, while the motion remains irrotational throughout the bulk of the fluid. The irrotational motion, however, cannot in general satisfy the condition of zero tangential stress at the free surface. Therefore the non-zero irrotational tangential stress near the surface drags a thin vortical layer along, making a modification to the velocity field. For a drop oscillating in a tenuous gas of negligible dynamic viscosity, viscous effects are extremely weak and they can be formulated from the potential-flow solution as a first approximation. This sort of argument can also be strengthened by the more rigorous calculations and detailed discussion in the papers of Kang & Leal (1987, 1988).

In this paper viscous effects are introduced through the normal damping stress N_d in (2.6), which equivalently represents the sum of the viscous pressure correction and normal visous stress in the more rigorous expressions used by Kang & Leal (1987, 1988). Instead of going through the analysis of the boundary-layer equations describing the weak vortical layer along the drop surface, however, the normal damping stress N_d is derived from the formula for estimating viscous dissipation of mechanical energy from the potential flow solution (Lamb 1932). Hence the mathematical system deals only with potential flow since no explicit analysis of the vortical layer is involved. Formally, the damping stress N_d can be mathematically determined if we equate the rate of overall work done by N_d to the total rate of dissipation of mechanical energy expressed in terms of a potential flow field (Lamb 1932):

$$\int_0^\pi N_d\left(F\frac{\partial \Phi}{\partial r_0}-\frac{1}{r_0}\frac{\partial F}{\partial \theta}\frac{\partial \Phi}{\partial \theta}\right)\sin\theta\,d\theta$$
$$=-\frac{\eta}{(\sigma\rho R)^{\frac{1}{2}}}\int_0^\pi\left(F\frac{\partial}{\partial r_0}-\frac{1}{r_0}\frac{\partial F}{\partial \theta}\frac{\partial}{\partial \theta}\right)\left[\left(\frac{\partial \Phi}{\partial r_0}\right)^2+\left(\frac{1}{r_0}\frac{\partial \Phi}{\partial \theta}\right)^2\right]\sin\theta\,d\theta, \quad (3.1)$$

where η is the viscosity of the drop, and the dimensionless parameter $\eta(\sigma\rho R)^{-\frac{1}{2}}$ may be regarded as the ratio of the timescale of viscous damping to the characteristic time for the oscillatory motion. It should be noted that when $\eta(\sigma\rho R)^{-\frac{1}{2}} \geqslant 0.1$, the results of the irrotational approximation used here differ somewhat from those of the normal-mode calculation which, according to Prosperetti (1980), represent the forced steady-state shape oscillations as the flow field is sufficiently developed. As $\eta(\sigma\rho R)^{-\frac{1}{2}}$ approaches zero, however, the irrotational and normal-mode results will become identical to each other (Reid 1960; Miller & Scriven 1968). Hence, the formulation given here should be reasonable for $\eta(\sigma\rho R)^{-\frac{1}{2}} \leqslant 0.1$.

In terms of the commonly used normal-mode analysis, N_d is written as

$$N_d=\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}N_d^{\langle n\rangle} \quad \text{with} \quad N_d^{\langle n\rangle}=\sum_{l=0}^{\infty}\gamma_l^{\langle n\rangle}(t)\,P_l(\theta). \quad (3.2)$$

Therefore, $\gamma_l^{\langle n\rangle}$ represents the damping stress coefficient for the lth mode corresponding to $P_l(\theta)$.

By means of the domain perturbation technique, we may expand each term in (3.1) at $r_0 = F(\theta, t)$ into corresponding terms at $r = 1$ with respect to the small parameter ϵ:

$$\left(F\frac{\partial \Phi}{\partial r_0}-\frac{1}{r_0}\frac{\partial F}{\partial \theta}\frac{\partial \Phi}{\partial \theta}\right)_{r_0=F(\theta,t)}=\left[\epsilon\frac{\partial \Phi^{\langle 1\rangle}}{\partial r}+O(\epsilon^2)\right]_{r=1} \quad (3.3)$$

and

$$\left(F\frac{\partial}{\partial r_0}-\frac{1}{r_0}\frac{\partial F}{\partial \theta}\frac{\partial}{\partial \theta}\right)\left[\left(\frac{\partial \Phi}{\partial r_0}\right)^2+\left(\frac{1}{r_0}\frac{\partial \Phi}{\partial \theta}\right)^2\right]_{r_0=F(\theta,t)}$$
$$=2\left\{\epsilon^2\left[\frac{\partial \Phi^{\langle 1\rangle}}{\partial r}\frac{\partial^2 \Phi^{\langle 1\rangle}}{\partial r^2}+\frac{\partial \Phi^{\langle 1\rangle}}{\partial \theta}\frac{\partial^2 \Phi^{\langle 1\rangle}}{\partial r\,\partial \theta}-\left(\frac{\partial \Phi^{\langle 1\rangle}}{\partial \theta}\right)^2\right]+O(\epsilon^3)\right\}_{r=1}. \quad (3.4)$$

Now, if we simply set

$$\frac{\eta}{(\sigma\rho R)^{\frac{1}{2}}} = \mu, \tag{3.5}$$

and consider any single mode with the velocity potential

$$\Phi^{\langle 1\rangle} = \beta_l^{\langle 1\rangle}(t)\, r^l P_l(\theta), \tag{3.6}$$

then substitution of (3.3) and (3.4) into (3.1) yields

$$l\beta_l^{\langle 1\rangle}\gamma_l^{\langle 1\rangle} = -2\mu l(l-1)(2+1)\,\beta_l^{\langle 1\rangle}\beta_l^{\langle 1\rangle}, \tag{3.7}$$

and thereby we have

$$\gamma_l^{\langle 1\rangle} = -2\mu(l-1)(2+1)\,\beta_l^{\langle 1\rangle}. \tag{3.8}$$

Relation (3.8) means that there should not be any explicit viscous effects in the absence of the velocity field.

4. Linear oscillations

Since the characteristics of linear drop oscillations are well known, it would be worthwhile to first examine the basic dynamical behaviour of our formulation in the linear expansion case. For the $O(\epsilon)$ problem, based on relation (2.17) and (3.5), the combination of the kinematic condition (2.5) and the equation of normal stress balance (2.6) leads to the following dynamical equations for the shape-function coefficients:

$$\frac{\partial^2\alpha_l^{\langle 1\rangle}}{\partial T_0^2} + 2\mu(2l+1)(l-1)\frac{\partial\alpha_l^{\langle 1\rangle}}{\partial T_0} + l(l-1)(l+2)\,\alpha_l^{\langle 1\rangle} = 0 \quad (l \neq 0,1,2) \tag{4.1}$$

and

$$\frac{\partial^2\alpha_2^{\langle 1\rangle}}{\partial T_0^2} + 10\mu\frac{\partial\alpha_2^{\langle 1\rangle}}{\partial T_0} + 8\alpha_2^{\langle 1\rangle} = \tfrac{3}{2}K(1+\mathrm{e}^{\mathrm{i}2\Omega T_0}) + \text{c.c.}, \tag{4.2}$$

where c.c. stands for the complex conjugate of the preceding terms. The constraints (2.11) and (2.12) yield

$$\alpha_0^{\langle 1\rangle} = \alpha_1^{\langle 1\rangle} = 0, \quad \Delta p_0^{\langle 1\rangle} = -\tfrac{3}{4}K(1+\cos 2\Omega T_0). \tag{4.3}$$

Equation (4.1) describes free linear oscillators with linear damping, and has the general solution

$$\alpha_l^{\langle 1\rangle} = c_l^{\langle 1\rangle}\,\mathrm{e}^{\lambda_l T_0} + \text{c.c.} \tag{4.4}$$

where $c_l^{\langle 1\rangle}$ can be functions of slower timescales $T_1, T_2, \ldots$, and

$$\lambda_l = -\mu(2l+1)(l-1) \pm [\mu^2(2l+1)^2(l-1)^2 - l(l-1)(l+2)]^{\frac{1}{2}}. \tag{4.5}$$

Therefore, in dimensionless form, Lamb's (1932) result for the decay time of the lth free mode corresponds to $[\mu(2l+1)(l-1)]^{-1}$ and, when $\mu \to 0$, Rayleigh's (1879) characteristic frequencies are recovered as $[l(l-1)(l+2)]^{\frac{1}{2}}$. Non-zero μ results in a frequency lowering. Moreover, for a given μ, aperiodic motion is possible as l exceeds some value such that $[l(l-1)(l+2)]^{\frac{1}{2}}[(2l+1)(l-1)]^{-1} \leqslant \mu$.

Equation (4.2) represents a linear oscillator with linear damping and a single harmonic force. Besides the homogeneous solution given by (4.4), it has a particular solution of the form

$$\tfrac{3}{16}K + A_0 K\frac{2-\Omega^2-\mathrm{i}5\mu\Omega}{[(2-\Omega^2)^2+25\mu^2\Omega^2]^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}2\Omega T_0} + \text{c.c.}, \tag{4.6}$$

where
$$\Lambda_0 = \frac{3}{8[(2-\Omega^2)^2+25\mu^2\Omega^2]^{\frac{1}{2}}}.$$

Hence, a primary resonance will occur when $\Omega \approx \frac{1}{2}\omega_2$ since the excitation force in (4.2) varies at twice the frequency of the electric field. As might also be noted, besides exciting a two-lobed oscillatory mode, the pure a.c. external electric field also produces a quiescent prolate distortion of the drop surface (represented by the first term in (4.6)). This quiescent deformation is expected to lower the characteristic frequencies (Feng 1990; Feng & Beard 1990).

It should be pointed out that the statements for the ordering of the electric stress (2.17) and damping stress (3.5) in the perturbation calculations do not necessarily mean that those constants of proportionality should satisfy $K = O(1)$ and $\mu = O(1)$. Rather, for the perturbation solutions to be valid, we should scale the shape deformation properly, i.e. $F^{\langle n\rangle} = O(1)$ while $\epsilon \ll 1$. For example, according to (4.6), the resonant peak value for the oscillation amplitude $2\Lambda_0 K$ at $\Omega \approx \frac{1}{2}\omega_2$ reads as $3K/(10\mu\omega_2)$. Thus for $2\Lambda_0 K = 1$, we have the relation

$$K = 9.43\mu, \tag{4.7}$$

so we could have $K = O(1)$ while $\mu = O(10^{-1})$ or $K = O(10^{-1})$ for $\mu = O(10^{-2})$, etc. to keep $F^{\langle 1\rangle} = O(1)$. However, for a given ϵ, the value for K should be confined within the range where drops are stable. As a point of reference, a water drop in air with radius 2.5 mm has $\mu = 0.00235$ ($\ll 0.1$), so for $\epsilon = 0.1$ we need $K = 0.0222$, thus $E^*_{\mathrm{a}} \approx 0.85$ kV/cm, to satisfy the relation (4.7).

Furthermore, from the equations for the conservation of electric charge and for the continuity of the tangential component of the electric field, the electric potential at this level of approximation can be determined as

$$\xi_l^{\langle 1\rangle} = 3\cos\Omega T_0\left(\frac{l}{2l-1}\alpha_{l-1}^{\langle 1\rangle} + \frac{l+1}{2l+3}\alpha_{l+1}^{\langle 1\rangle}\right). \tag{4.8}$$

5. Secondary resonances

According to the linear analysis in the last section, the alternating uniform electric field only excites a two-lobed shape oscillation owing to the constraint of the spatial form of the first-order electric surface stress and, when Ω is away from $\frac{1}{2}\omega_2$, the effect of the excitation will be small. For a nonlinear system, however, secondary resonances may occur under strong (hard) external excitation even if 2Ω is away from its characteristic frequencies (Nayfeh & Mook 1979), and other modes may also be excited by the alternating uniform electric field through nonlinear mode interactions. In this section, a second-order expansion of domain perturbations is carried out and various secondary resonances are investigated.

In order to maintain the condition of strong excitation so that the amplitude of the external forcing term is one order greater than that of viscous effects, it is mathematically assumed that

$$\frac{\eta}{(\sigma\rho R)^{\frac{1}{2}}} = \epsilon\mu, \tag{5.1}$$

rather than (3.5). Relation (5.1) apparently means that we are only dealing with the situation where the timescale for viscous dissipation, or equivalently of vorticity diffusion from the interface, is much longer than the characteristic time for the oscillatory motion. Thus, instead of (3.8) we get

$$N_{\mathrm{d}}^{\langle 1\rangle} = 0, \quad \text{while} \quad \gamma_l^{\langle 2\rangle} = -4\mu(l-1)(2l+1)\beta_l^{\langle 1\rangle}; \tag{5.2}$$

viscous effects first explicitly appear in the second-order problem. The first-order solution thereby takes the form (rather than that given by (4.4)–(4.6))

$$\alpha_0^{\langle 1\rangle} = \alpha_1^{\langle 1\rangle} = 0, \quad \alpha_l^{\langle 1\rangle} = c_l^{\langle 1\rangle}\,\mathrm{e}^{\mathrm{i}\omega_l T_0} + \text{c.c.}, \tag{5.3a}$$

and
$$\alpha_2^{\langle 1\rangle} = c_2^{\langle 1\rangle}\,\mathrm{e}^{\mathrm{i}\omega_2 T_0} + \tfrac{3}{16}K + \Lambda K\,\mathrm{e}^{\mathrm{i}2\Omega T_0} + \text{c.c.}, \quad \text{where} \quad \Lambda = \frac{3}{8(2-\Omega^2)}, \tag{5.3b}$$

with
$$\omega_l^2 = l(l-1)(l+2). \tag{5.4}$$

The free oscillation terms in (5.3) do not explicitly show the viscous decay factor as expressed in (4.4) and (4.5). This does not mean that there are no such viscous damping effects. The viscous effects are actually contained in the coefficients $c_l^{\langle 1\rangle}$, which are presumed to be functions of slower timescales $T_1, T_2, \ldots$. Thus the mathematical forms in (5.3) indeed express the physical meaning of the assumption (5.1) of a slower timescale for viscous damping in comparison with that for oscillatory motions. As will be seen, the viscous decay factor can be shown to be related to T_1 in the second-order problem.

The second-order expansion of the domain perturbations for the kinematic condition yields

$$\frac{\partial \alpha_l^{\langle 2\rangle}}{\partial T_0} = l\beta_l^{\langle 2\rangle} - 2\frac{\partial \alpha_l^{\langle 1\rangle}}{\partial T_1} - (2l+1)\sum_{j,k}\left[\frac{1}{k}\left\langle \frac{\mathrm{d}P_j}{\mathrm{d}\theta}\frac{\mathrm{d}P_k}{\mathrm{d}\theta}, P_l\right\rangle - (k-1)\langle P_j P_k, P_l\rangle\right]\alpha_j^{\langle 1\rangle}\frac{\partial \alpha_k^{\langle 1\rangle}}{\partial T_0}, \tag{5.5}$$

and the condition of normal stress balance leads to

$$\begin{aligned}
\frac{\partial \beta_l^{\langle 2\rangle}}{\partial T_0} &+ (l-1)(l+2)\alpha_l^{\langle 2\rangle} + \delta_{l0}\Delta p_0^{\langle 2\rangle} = -2\frac{1}{l}\frac{\partial^2 \alpha_l^{\langle 1\rangle}}{\partial T_1 \partial T_0} \\
&+ 2(2l+1)\sum_{j,k}(k^2+k-1)\langle P_j P_k, P_l\rangle \alpha_j^{\langle 1\rangle}\alpha_k^{\langle 1\rangle} - (2l+1)\sum_{j,k}\langle P_j P_k, P_l\rangle \alpha_j^{\langle 1\rangle}\frac{\partial^2 \alpha_k^{\langle 1\rangle}}{\partial T_0^2} \\
&- \tfrac{1}{2}(2l+1)\sum_{j,k}\left(\langle P_j P_k, P_l\rangle + \frac{1}{jk}\left\langle \frac{\mathrm{d}P_j}{\mathrm{d}\theta}\frac{\mathrm{d}P_k}{\mathrm{d}\theta}, P_l\right\rangle\right)\frac{\partial \alpha_j^{\langle 1\rangle}}{\partial T_0}\frac{\partial \alpha_k^{\langle 1\rangle}}{\partial T_0} \\
&+ 18K\cos^2\Omega T_0\left\{lI_{-2}(l+2)\,\alpha_{l+2}^{\langle 1\rangle} + l\left[I_0(l) - \frac{2l}{(2l+1)(2l-1)}\right]\alpha_l^{\langle 1\rangle} + (l-2)I_{+2}(l-2)\,\alpha_{l-2}^{\langle 1\rangle}\right\} \\
&- 4\mu\frac{(2l+1)(l-1)}{l}\frac{\partial \alpha_l^{\langle 1\rangle}}{\partial T_0},
\end{aligned} \tag{5.6}$$

where the symbol $\langle\,,\rangle$ stands for the inner product of the functions weighted with $\sin\theta$ on the interval $[\theta = 0, \theta = \pi]$, and

$$I_{-2}(l) = \frac{l(l-1)}{(2l-1)(2l+1)}, \quad I_0(l) = \frac{2l^2+2l-1}{(2l-1)(2l+3)}, \quad I_{+2}(l) = \frac{(l+1)(l+2)}{(2l+1)(2l+3)}.$$

In general, there appears to be an infinite number of combinations of modes j and k in (5.5) and (5.6), so the second-order problem becomes the solution for a system having infinite degrees of freedom. However, most first-order amplitudes of the jth and kth modes ($c_j^{\langle 1\rangle}$ and $c_k^{\langle 1\rangle}$) will decay owing to viscous effects. There can only be a few non-decaying resonant modes maintained by the external force with a properly tuned frequency. In mathematical terms, the first-order amplitude for the lth mode will always decay exponentially (see (5.9)), provided that the inhomogeneous terms associated with the forced oscillation term in the expression for $\alpha_2^{\langle 1\rangle}$ (equation (5.3b)), which has a frequency of 2Ω, do not produce secular and small-divisor terms in the equation for the lth mode or in the equation for a mode that has a characteristic frequency of double, one half or one third of ω_l. The contributions from those exponentially decaying first-order modes become negligibly small with time. In most cases, it is only necessary to consider one first-order mode which has a non-decaying amplitude. When some coupled resonances occur, however, two coupling first-order modes must be considered together. It seems impossible to have more than two first-order modes excited together by a single harmonic forcing term at the level of a second-order approximation with a quadratic nonlinearity.

5.1. *Secondary resonances of the two-lobed mode*

We start with the simplest case where only a two-lobed mode is present in the first-order solution, i.e.

$$F^{\langle 1\rangle} = \alpha_2^{\langle 1\rangle} P_2, \quad \text{where} \quad \alpha_2^{\langle 1\rangle} = c_2^{\langle 1\rangle} \mathrm{e}^{\mathrm{i}\omega_2 T_0} + \tfrac{3}{16}K + \Lambda K \mathrm{e}^{\mathrm{i}2\Omega T_0} + \text{c.c.} \tag{5.7}$$

The dynamical equation for the shape-function coefficient $\alpha_2^{\langle 2\rangle}$ can be obtained by eliminating $\beta_2^{\langle 2\rangle}$ from (5.5) and (5.6):

$$\begin{aligned} \frac{\partial^2 \alpha_2^{\langle 2\rangle}}{\partial T_0^2} + 8\alpha_2^{\langle 2\rangle} = & -4\frac{\partial^2 \alpha_2^{\langle 1\rangle}}{\partial T_1 \partial T_0} - 20\mu \frac{\partial \alpha_2^{\langle 1\rangle}}{\partial T_0} + 100\langle P_2 P_2, P_2\rangle \alpha_2^{\langle 1\rangle}\alpha_2^{\langle 1\rangle} \\ & -10\langle P_2, P_2, P_2\rangle \alpha_2^{\langle 1\rangle} \frac{\partial^2 \alpha_2^{\langle 1\rangle}}{\partial T_0^2} - 5\left(\langle P_2 P_2, P_2\rangle + \frac{1}{4}\left\langle \frac{\mathrm{d}P_2}{\mathrm{d}\theta}\frac{\mathrm{d}P_2}{\mathrm{d}\theta}, P_2\right\rangle\right)\frac{\partial \alpha_2^{\langle 1\rangle}}{\partial T_0}\frac{\partial \alpha_2^{\langle 1\rangle}}{\partial T_0} \\ & -5\left[\frac{1}{2}\left\langle \frac{\mathrm{d}P_2}{\mathrm{d}\theta}\frac{\mathrm{d}P_2}{\mathrm{d}\theta}, P_2\right\rangle - \langle P_2 P_2, P_2\rangle\right]\left(\alpha_2^{\langle 1\rangle}\frac{\partial^2 \alpha_2^{\langle 1\rangle}}{\partial T_0^2} + \frac{\partial \alpha_2^{\langle 1\rangle}}{\partial T_0}\frac{\partial \alpha_2^{\langle 1\rangle}}{\partial T_0}\right) \\ & + \tfrac{162}{35}K(1 + \mathrm{e}^{\mathrm{i}2\Omega T_0} + \text{c.c.})\,\alpha_2^{\langle 1\rangle}. \end{aligned} \tag{5.8}$$

By substituting (5.7) into (5.8), it is seen that secondary resonances may occur when $\Omega \approx 0$, $\Omega \approx \frac{1}{4}\omega_2$ and $\Omega \approx \omega_2$. Primary resonance also takes place when $\Omega \approx \frac{1}{2}\omega_2$, the first approximation for which has been given in §4, where the ordering of the magnitude of viscous effects is given by (3.5) rather than (5.1) to keep the resonant amplitude bounded. There is, however, an alternative way to calculate the case of primary resonance, for which the ordering form (5.1) is kept but (2.17) is replaced by $\epsilon_\mathrm{m} R E_\mathrm{a}^{*2}/\sigma = \epsilon^2 K$. Hence, the expression for the first-order primary resonant response (4.6) can also be determined from the solvability condition for the second-order expansion equation. Second-order corrections to the primary resonance case will not be pursued in this paper, because it takes tremendous efforts to determine the higher-order viscous stress and, on the other hand, its effects are expected to be relatively insignificant. If Ω is away from all these resonant values, the solvability condition for (5.8) would be

$$\frac{\partial c_2^{\langle 1\rangle}}{\partial T_1} + \left(5\mu + \mathrm{i}\frac{387}{70}\frac{K}{\omega_2}\right)c_2^{\langle 1\rangle} = 0, \tag{5.9}$$

which yields an exponential decay factor as well as frequency lowering for the first-order free oscillation term $c_2^{\langle 1\rangle}\mathrm{e}^{\mathrm{i}\omega_2 T_0} + \text{c.c.}$ in (5.7). Hence eventually only forced oscillation (terms associated with $AK\,\mathrm{e}^{\mathrm{i}2\Omega T_0}+\text{c.c.}$) can be observed while the first-order free oscillations will be damped out, if there are no resonances occurring. It would not be difficult to generally show that all first-order modes would manifest similar behaviour if Ω does not have the value that can cause resonances. Thus, if the excitation frequency Ω is away from the values capable of causing any resonances, all of the oscillation modes forced through the nonlinear coupling with the first-order forced two-lobed mode ($AK\,\mathrm{e}^{\mathrm{i}2\Omega T_0}+\text{c.c.}$ in (5.7)) can only have an amplitude of second order in ϵ. The practical effects of the oscillations with second-order amplitudes are expected to be of much less significance, so the calculations for the non-resonant cases are omitted here. Since the natural periodicity of the drop's surface results in a discrete spectrum of the fundamental modes, it would be rare to have other modes in resonances when the external forcing is tuned to excite one particular mode. So the validity of assumption (5.7) for the study of most secondary resonances of the two-lobed mode might be justified. When we study the resonance for the case $\Omega \approx \omega_2$ in §5.1.3, however, the four-lobed mode must be included because $3\omega_2 = \omega_4$ and thereby coupled resonance occurs.

5.1.1. *Slowly varying excitation* ($\Omega \approx 0$)

Since in this case $\cos 2\Omega t$ is slowly varying, 2Ω is written as $\epsilon\Omega_\mathrm{d}$. Then

$$\mathrm{e}^{\mathrm{i}2\Omega T_0} = \mathrm{e}^{\mathrm{i}\Omega_\mathrm{d} T_1}. \tag{5.10}$$

The solvability condition for (5.8) would be

$$\frac{\partial c_2^{\langle 1\rangle}}{\partial T_1} + \left(5\mu + \mathrm{i}\frac{387}{70}\frac{K}{\omega_2}\right)c_2^{\langle 1\rangle} + \mathrm{i}\frac{387}{140}\frac{K}{\omega_2}c_2^{\langle 1\rangle}(\mathrm{e}^{\mathrm{i}\Omega_\mathrm{d} T_1} + \mathrm{e}^{-\mathrm{i}\Omega_\mathrm{d} T_1}) = 0. \tag{5.11}$$

Therefore we have

$$c_2^{\langle 1\rangle} = c_2^{\langle 2\rangle}\exp\left[-\left(5\mu + \mathrm{i}\frac{387}{70}\frac{K}{\omega_2}\right)T_1 - \mathrm{i}\frac{387K}{70\omega_2\Omega_\mathrm{d}}\sin\Omega_\mathrm{d} T_1\right], \tag{5.12}$$

where $c_2^{\langle 2\rangle}$ could be a function of slower timescales $T_2, T_3, \ldots$. We see that the slowly varying external electric field cannot maintain oscillatory motions.

5.1.2. *Superharmonic resonances* ($\Omega \approx \frac{1}{4}\omega_2$)

To express the proximity of Ω to $\frac{1}{4}\omega_2$, a detuning parameter Ω_d is introduced by defining

$$4\Omega = \omega_2 + \epsilon\Omega_\mathrm{d}. \tag{5.13}$$

Thus

$$4\Omega T_0 = \omega_2 T_0 + \Omega_\mathrm{d} T_1. \tag{5.14}$$

Then the solvability condition for (5.8) would be

$$\frac{\partial c_2^{\langle 1\rangle}}{\partial T_1} + (5\mu + \mathrm{i}\Delta\omega_2 K)\,c_2^{\langle 1\rangle} + \mathrm{i}\frac{A^2K^2}{14\omega_2}\left(40 + 38\Omega^2 + \frac{81}{5A}\right)\mathrm{e}^{\mathrm{i}\Omega_\mathrm{d} T_1} = 0, \tag{5.15}$$

where we define

$$\Delta\omega_2 \equiv \frac{387}{70\omega_2}.$$

The solution to (5.15) is found to be

$$c_2^{\langle 1\rangle} = c_2^{\langle 2\rangle}\,\mathrm{e}^{-(5\mu + \mathrm{i}\Delta\omega_2 K)\,T_1} - \tfrac{1}{2}A_\mathrm{s}K^2\frac{(\Omega_\mathrm{d} + \Delta\omega_2 K + \mathrm{i}5\mu)}{[(\Omega_\mathrm{d} + \Delta\omega_2 K)^2 + 25\mu^2]^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\Omega_\mathrm{d} T_1}, \tag{5.16}$$

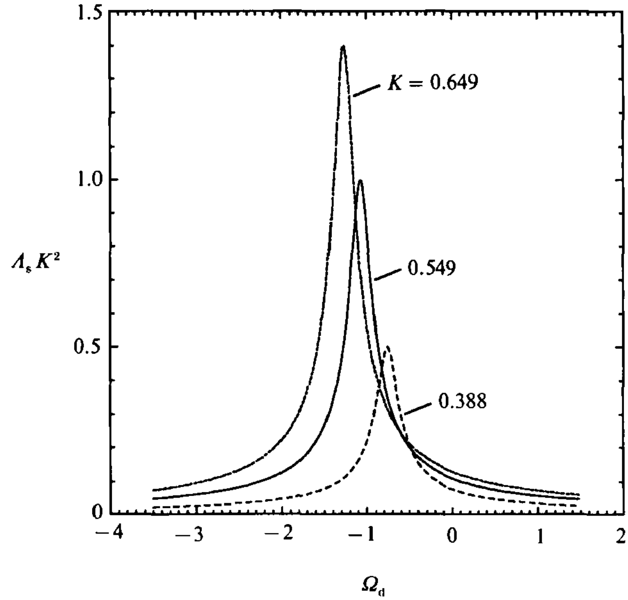

FIGURE 1. Frequency-response curves for superharmonic resonances with $\Omega_d = 0$ corresponding to $\Omega = \frac{1}{4}\omega_2$.

where
$$A_s = \frac{A^2}{7\omega_2}\frac{40+38\Omega^2+81/(5A)}{[(\Omega_d+\Delta\omega_2 K)^2+25\mu^2]^{\frac{1}{2}}}.$$

Therefore to a first approximation, the steady-state solution (as $t \rightarrow \infty$) is expected to be
$$F^{\langle 1\rangle} = [\tfrac{3}{8}K+2AK\cos 2\Omega t+A_s K^2 \sin(4\Omega t-\Theta)]P_2(\theta), \tag{5.17}$$
where $\Theta = \tan^{-1}((\Omega_d+\Delta\omega_2 K)/5\mu)$. As indicated in figure 1, the superharmonic resonant peak appears at $\Omega_d = -\Delta\omega_2 K$ instead of $\Omega_d = 0$ owing to the frequency shift resulting from the quiescent deformation of the interface shown in (4.6). The enhancement of such a frequency shift as K increases is shown by the frequency-response curves with different values for K.

Using the same point of reference as in §4 (2.5 mm radius water drop in air), $E_a^* \approx 4.2$ kV/cm will be needed to excite oscillations at $\Omega = \frac{1}{4}(\omega_2-\epsilon\Delta\omega_2 K)$ with $\epsilon = 0.1$ and $A_s K^2 = 1$. Hence, in order to have the same amplitude of oscillation, excitation through secondary resonances needs a stronger external field than that needed by the primary resonance (where $E_a^* \approx 0.85$ kV/cm is required). At the same excitation frequency for superharmonic resonance, the solution of the linear problem (the second term in (5.17)) renders only about 27% of the amplitude given by the third term. Since the third term in (5.17), corresponding to superharmonic resonance, is proportional to K^2, it will however disappear more rapidly than the second term as K vanishes, and this solution approaches the solution of the linear problem.

The scaling relations (3.5) and (5.1) apparently require different magnitudes of viscous effects for the problems of primary and secondary resonances. In reality, however, as long as the viscous effects are relatively weak ($\eta(\sigma\rho R)^{-\frac{1}{2}} \ll 0.1$) so that the treatment in §3 should be reasonable, both (3.5) and (5.1) can be applied to the same physical system as shown in our examples for a 2.5 mm water drop in air. The small parameter ϵ is a mathematical device only for book-keeping in perturbation manipulations, whereas for a drop with a given viscosity the magnitude of oscillations is controlled in practice by changing the electric field intensity (E_a^*) and excitation frequency (Ω). In other words, for an arbitrarily given small parameter ϵ and the physical properties (η, σ, ρ and R) of the drop in question, μ can be determined either from the relation (3.5) or (5.1) depending on whether the problem is primary or secondary resonances. Then, the magnitude of oscillations is calculated from the applied electric field intensity E_a^* through K and the excitation frequency Ω.

Qualitatively similar results of the superharmonic resonance studied here will be expected for the case when $\Omega \approx \frac{1}{4}\omega_4 = \frac{3}{4}\omega_2$; in this instance the equation for the four-lobed mode will have terms of frequency 4Ω that will lead to small-divisor terms. This actually provides a way to excite marked four-lobed shape oscillations with a uniform electric field, which only excites two-lobed shape oscillations according to linear analysis.

5.1.3. *Coupled resonance of two- and four-lobed modes* ($\Omega \approx \omega_2$)

Since in this case simultaneous resonance for both two- and four-lobed modes occurs, in addition to (5.7) we must also include a four-lobed mode to the first-order solution. Therefore,

$$F^{\langle 1\rangle} = \alpha_2^{\langle 1\rangle}P_2+\alpha_4^{\langle 1\rangle}P_4 = (c_2^{\langle 1\rangle}\,\mathrm{e}^{\mathrm{i}\omega_2 T_0}+\tfrac{3}{16}K+AK\,\mathrm{e}^{\mathrm{i}2\Omega T_0}+\text{c.c.})P_2+(c_4^{\langle 1\rangle}\,\mathrm{e}^{\mathrm{i}\omega_4 T_0}+\text{c.c.})P_4. \tag{5.18}$$

To analyse this kind of coupled resonances to first order, we let

$$\Omega = \omega_2+\epsilon\Omega_\mathrm{d},$$

so that
$$(2\Omega-\omega_2)\,T_0 = \omega_2\,T_0+2\Omega_\mathrm{d}\,T_1,\quad (\omega_4-2\Omega)\,T_0 = \omega_2\,T_0-2\Omega_\mathrm{d}\,T_1. \tag{5.19}$$

The dynamical equations for the two coupled modes yield solvability conditions of the forms

$$\frac{\partial c_2^{\langle 1\rangle}}{\partial T_1}+(5\mu+\mathrm{i}\Delta\omega_2 K)\,c_2^{\langle 1\rangle}+\mathrm{i}C_{22}K\overline{c_2^{\langle 1\rangle}}\,\mathrm{e}^{\mathrm{i}2\Omega_\mathrm{d}T_1}+\mathrm{i}C_{24}Kc_4^{\langle 1\rangle}\,\mathrm{e}^{-\mathrm{i}2\Omega_\mathrm{d}T_1} = 0 \tag{5.20a}$$

and
$$\frac{\partial c_4^{\langle 1\rangle}}{\partial T_1}+(27\mu+\mathrm{i}\Delta\omega_4 K)\,c_4^{\langle 1\rangle}+\mathrm{i}C_{42}Kc_2^{\langle 1\rangle}\,\mathrm{e}^{\mathrm{i}2\Omega_\mathrm{d}T_1} = 0, \tag{5.20b}$$

where
$$C_{22} = \frac{1}{28\omega_2}[A(160+14\omega_2^2)+\tfrac{162}{5}] \approx 0.194,$$

$$C_{24} = \frac{1}{28\omega_2}[A(384+6\omega_4^2+96\omega_2^2-64\omega_2\,\omega_4)+24] \approx 0.265,$$

$$C_{42} = \frac{1}{140\omega_4}[A(2880-252\omega_2^2)+432] \approx 0.318,$$

and
$$\Delta\omega_4 \equiv \frac{4489}{154\omega_4}.$$

Furthermore we let

$$c_2^{\langle 1\rangle} = A\,\mathrm{e}^{\mathrm{i}\Omega_\mathrm{d}T_1}\quad\text{and}\quad c_4^{\langle 1\rangle} = B\,\mathrm{e}^{\mathrm{i}3\Omega_\mathrm{d}T_1}$$

so that (5.20 a, b) lead to

$$\frac{\partial A_r}{\partial T_1}+5\mu A_r-(\Omega_d+\Delta\omega_2 K-C_{22}K)A_i-C_{24}KB_i=0, \tag{5.21 a}$$

$$\frac{\partial A_i}{\partial T_1}+5\mu A_i+(\Omega_d+\Delta\omega_2 K+C_{22}K)A_r+C_{24}KB_r=0, \tag{5.21 b}$$

$$\frac{\partial B_r}{\partial T_1}+27\mu B_r-(3\Omega_d+\Delta\omega_4 K)B_i-C_{42}KA_i=0, \tag{5.21 c}$$

$$\frac{\partial B_i}{\partial T_1}+27\mu B_i+(3\Omega_d+\Delta\omega_4 K)B_r+C_{42}KA_r=0; \tag{5.21 d}$$

where we put $$A=A_r+\mathrm{i}A_i \quad\text{and}\quad B=B_r+\mathrm{i}B_i.$$

Equations (5.21 a–d) are a set of linear equations having constant coefficients, so the solution can be expressed in the form

$$(A_r,A_i,B_r,B_i)=(a_r,a_i,b_r,b_i)\,\mathrm{e}^{\lambda K T_1}, \tag{5.22}$$

where a_r, a_i, b_r, b_i and λ are constants. Upon substituting (5.22) into (5.21), it is found that in order to have a non-trivial solution, λ must be the eigenvalues of the matrix

$$\begin{bmatrix} -5\hat{\mu} & \hat{\Omega}_d+\Delta\omega_2-C_{22} & 0 & C_{24} \\ -(\hat{\Omega}_d+\Delta\omega_2+C_{22}) & -5\hat{\mu} & -C_{24} & 0 \\ 0 & C_{42} & -27\hat{\mu} & 3\hat{\Omega}_d+\Delta\omega_4 \\ -C_{42} & 0 & -(3\hat{\Omega}_d+\Delta\omega_4) & -27\hat{\mu} \end{bmatrix}, \tag{5.23}$$

where we define $$\hat{\mu}\equiv\frac{\mu}{K},\quad \hat{\Omega}_d\equiv\frac{\Omega_d}{K}.$$

The characteristic equation for (5.23) is a quartic equation. When all the real parts of the four values of λ are negative, the oscillation terms associated with $c_2^{\langle 1\rangle}$ and $c_4^{\langle 1\rangle}$ will decay exponentially. For the motions of the resonances to become noticeably large, at least one of the real parts of the four eigenvalues should be greater than zero. A positive real part of eigenvalues λ does not necessarily imply exponentially growing amplitudes without bound. Actually, in many cases the amplitudes have been shown to have finite values when numerical calculations of the primitive nonlinear equations are carried out (Nayfeh & Mook 1979) or the solvability condition for the third-order problem is studied (Nayfeh 1983). Although the present perturbation analysis cannot provide detailed information about the large-amplitude motions, it clearly points to the conditions for the presence and absence of resonant oscillations.

Shaded areas in figure 2 indicate the regions in the $(\hat{\Omega}_d,\hat{\mu})$-plane where the maximum real part of the eigenvalues is positive. It is obvious that large-amplitude oscillations will probably occur when $\hat{\Omega}_d\approx-\Delta\omega_2$, where the frequency of excitation is tuned to match the shifted frequency for the two-lobed mode caused by the quiescent deformation. When viscous effects are extremely small in comparison with the excitation field intensity, two-degree-of-freedom coupling results in two additional parameter ranges where large-amplitude oscillations might be observed: one appears when $\hat{\Omega}_d\approx-\frac{1}{4}(\Delta\omega_2+\Delta\omega_4)$ and another close to $3\hat{\Omega}_d\approx-\Delta\omega_4$. Taking

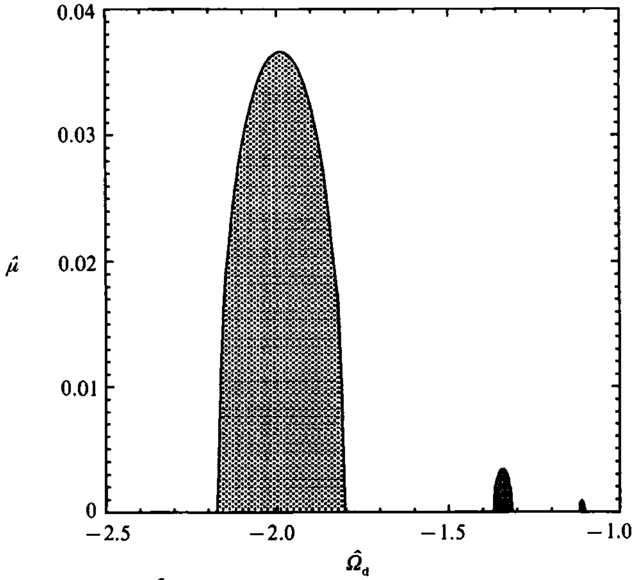

FIGURE 2. Regions in the $(\hat{\Omega}_d, \hat{\mu})$-parameter plane for the exponentially growing (shaded) and decaying (unshaded) amplitudes of the coupled resonances as $\Omega \approx \omega_2$.

$\hat{\mu} = 0.0366$, which is about the maximum value possible for λ to have a positive real part, $E_a^* \approx 4.6\,\mathrm{kV/cm}$ will be needed for the excitation of large-amplitude oscillations, for a water drop in air with radius 2.5 mm.

5.2. *Subharmonic resonance of the third mode* ($\Omega \approx \omega_3$)

By virtue of the quadratic terms in the second-order equations (5.5) and (5.6), every lth mode can be excited at least as $\Omega \approx \omega_l$. If there are no other modes internally interacting with this lth mode in resonance, the problem becomes a single-degree-of-freedom subharmonic resonance. As an example of such a subharmonic resonance of a single mode, we solve for the case of $\Omega \approx \omega_3$ in this subsection. Similar behaviour for the case of $\Omega \approx \omega_4$ and other single-mode excitations should be expected.

The detuning parameter here is defined by

$$(2\Omega - \omega_3 T_0) = \omega_3 T_0 + 2\Omega_d T_1. \tag{5.24}$$

The solvability condition for the third mode takes the form

$$\frac{\partial c_3^{\langle 1\rangle}}{\partial T_1} + (14\mu + i\Delta\omega_3 K)\, c_3^{\langle 1\rangle} + iC_{32} K \overline{c_3^{\langle 1\rangle}}\, e^{i2\Omega_d T_1} = 0, \tag{5.25}$$

where
$$C_{32} = \frac{1}{20\omega_3}\{A[256 + \tfrac{112}{3}\Omega^2 + \tfrac{16}{3}\omega_3^2 - \tfrac{64}{3}\Omega\omega_3] + \tfrac{963}{14}\} \approx 0.518$$

and
$$\Delta\omega_3 \equiv \frac{411}{28\omega_3}.$$

If we let

$$c_3^{\langle 1\rangle} = A\, e^{i\Omega_d T_1} \quad \text{with} \quad A = A_r + iA_i, \quad (A_r, A_i) = (a_r, a_i)\, e^{\lambda K T_1}, \tag{5.26}$$

in order to have a non-trivial solution we must have

$$\lambda = -14\hat{\mu} \pm [C_{32}^2 - (\hat{\Omega}_d + \Delta\omega_3)^2]^{\frac{1}{2}}. \tag{5.27}$$

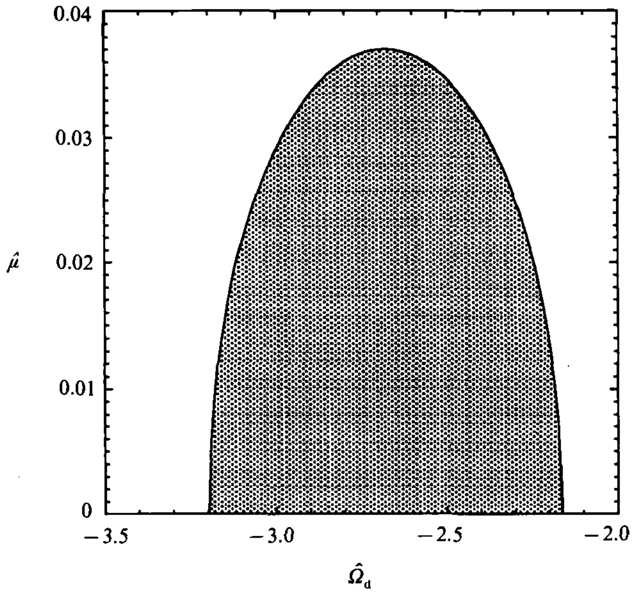

FIGURE 3. Regions in the $(\hat{\Omega}_d, \hat{\mu})$-parameter plane for the exponentially growing (shaded) and decaying (unshaded) amplitude of the subharmonic third-mode resonances as $\Omega \approx \omega_3$.

Hence the condition for A to become large is that $(\hat{\Omega}_d + \Delta\omega_3)^2 \leqslant C_{32}^2 - 196\hat{\mu}^2$. When $\hat{\mu} > \frac{1}{14}C_{32}$, A always decays exponentially. For the excitation of a single mode such as the case of Ω near ω_3, figure 3 shows that in the $(\hat{\Omega}_d, \hat{\mu})$-plane there is only one region where large-amplitude oscillation may occur, unlike the case of excitation of two coupled modes where three regions may appear, as seen in figure 2. For a water drop in air with radius 2.5 mm, $E_a^* \approx 4.6$ kV/cm will be needed for the excitation of large-amplitude subharmonic oscillations of the three-lobed mode at $\Omega \approx \omega_3$.

6. Conclusions

The perturbation analysis of nonlinear dynamical equations for a conducting drop with slight viscous effects in an alternating electric field provides basic insight into the secondary resonances occurring when both the frequency and spatial form of the excitation do not directly match the characteristic frequencies and the mode shapes of the drop. For the two-lobed mode, large-amplitude oscillations are shown to be possible when the frequency of external excitation is close to $\frac{1}{2}\omega_2$ (superharmonic resonance) and $2\omega_2$ (coupled resonance) due to the quadratic nonlinearity in the second-order problem. If the excitation frequency is near $2\omega_2$, inducing the resonance of the two-lobed mode, large-amplitude oscillation of the four-lobed mode will also be excited because $\omega_4 = 3\omega_2$ and thereby $2\Omega + \omega_2 \approx \omega_4$ when $2\Omega - \omega_2 \approx \omega_2$. This kind of coupling of two modes is unique in forced oscillations, whereas it appears in a free oscillation system only when the third-order expansion is carried out, which presents a cubic nonlinearity (Tsamopoulos & Brown 1984; Natarajan & Brown 1987). Large (first-order) amplitude oscillations of other modes are also found in studying the second-order problem when $2\Omega - \omega_l \approx \omega_l$ (subharmonic resonances), even though the spatial form of the first-order electric stress only coincides with the two-lobed mode.

According to the results obtained in this paper, when the electric field intensity is large enough, in addition to the primary band obtained from the linear analysis there

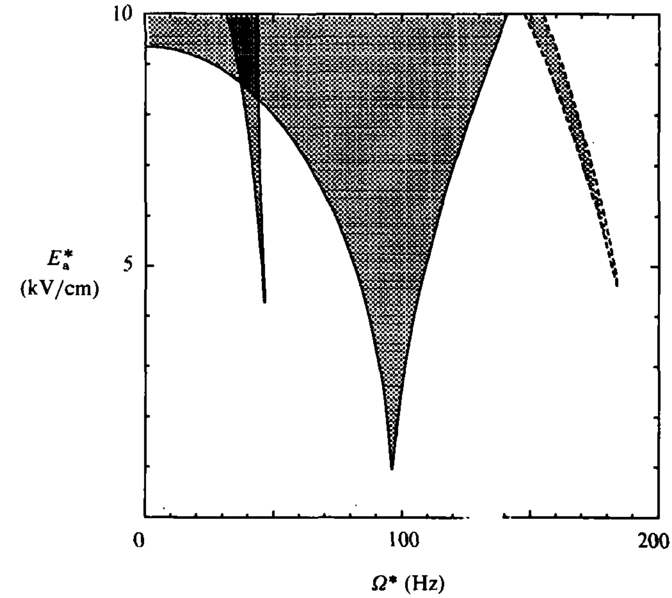

FIGURE 4. The (Ω^*, E_a^*)-plane showing the parameter regions (shaded) for the response oscillation amplitudes of the two-lobed mode greater than one-tenth of the drop radius. Dimensional quantities are evaluated for a water drop with 2.5 mm radius in air.

are two small bands of the alternating frequency of the electric field, within which large-amplitude two-lobed mode oscillations are excited. This feature is shown in figure 4 for a water drop of 2.5 mm radius in air, where the shaded areas indicate the values for the electric field intensity E_a^* and frequency Ω^* to produce the two-lobed oscillation with amplitude greater than one-tenth of the drop radius. All quantities in figure 4 are shown in dimensional values denoted with asterisk. The primary band from linear analysis covers the largest area. For the coupled resonance, with the excitation frequency near $2\omega_2^*$, no quantitative information about the oscillation amplitudes is available from the second-order perturbation analysis. This band is contoured with a dashed line to show the region of exponential growth of initially small oscillation amplitudes. Both secondary resonance bands tilt towards lower frequencies as the electric field intensity increases owing to the shift of the drop's characteristic frequency by the quiescent deformation. The relatively small frequency shift for the primary band is not revealed by linear analysis. When $E_a^* \geqslant 8.5$ kV/cm, there is an area where the superharmonic band intersects the primary one. Thus the oscillation amplitudes for both the harmonics with frequencies around ω_2^* and $\frac{1}{2}\omega_2^*$ become noticeable. The areas of the small bands of secondary resonances are about an order smaller than that for the primary resonance, but with a properly tuned excitation frequency for the secondary resonance, the electric field intensity required for the excitation of large-amplitude oscillations is much less than that predicted by linear analysis.

Although the analysis in this paper deals with a specific physical problem of the oscillations of conducting drops excited by an alternating electric field, drop oscillations forced by other types of external excitations might exhibit similar phenomena, in view of the similar resonant behaviour found in other quadratic nonlinear systems where the physical forcing mechanisms are quite different from the present model (cf. Nayfeh & Mook 1979). Hence, the ubiquity of raindrop oscillations might be attributed to some secondary resonances with drop vortex shedding whose frequencies and spatial distributions, in general, differ from the characteristic frequencies and shape modes of the drops.

In designing an apparatus that can produce large-amplitude oscillations in the liquid drops one should take the secondary resonances into consideration, besides the well-known primary resonances. For instance, although the electrodes of certain geometry can produce an electric stress that matches only particular drop shape modes, other mode oscillations may also be excited, in the light of secondary resonance theory, by adjusting the voltage and frequency applied on the electrodes.

Moreover, for the secondary resonances to occur, the magnitudes of external excitations are required to be about an order greater than that for the primary resonances to produce comparable oscillation amplitudes. Hence, much stronger external forcing is expected to excite observable oscillations through other possible secondary resonances that may be revealed by carrying out third- or higher-order asymptotic expansions. As the quiescent shape of deformation is enhanced under a very strong external electric field, however, the drop may become unstable (Brazier-Smith *et al.* 1971). Hence, in practice there exists an upper limit for the magnitude of the alternating electric field that can be applied to excite drop oscillations.

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